

Binary nullity, Euler circuits and interlace polynomials

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Abstract

A theorem of Cohn and Lempel [J. Combin. Theory Ser. A **13** (1972), 83-89] gives an equality relating the number of circuits in a directed circuit partition of a 2-in, 2-out digraph to the $GF(2)$ -nullity of an associated matrix. This equality is essentially equivalent to the relationship between directed circuit partitions of 2-in, 2-out digraphs and vertex-nullity interlace polynomials of interlace graphs. We present an extension of the Cohn-Lempel equality that describes arbitrary circuit partitions in (undirected) 4-regular graphs. The extended equality incorporates topological results that have been of use in knot theory, and it implies that if H is obtained from an interlace graph by attaching loops at some vertices then the vertex-nullity interlace polynomial $q_N(H)$ is essentially the generating function for certain circuit partitions of an associated 4-regular graph.

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1 Introduction

Cohn and Lempel [9] gave a simple formula relating the number of orbits in a finite set under a certain kind of permutation to the nullity of an associated binary matrix. Let σ be the cyclic permutation $\sigma = (1\dots m)$ of the set $\{1, \dots, m\}$, let $\sigma_1, \dots, \sigma_k$ be pairwise disjoint transpositions of elements of $\{1, \dots, m\}$, and let $\pi = \sigma\sigma_1\dots\sigma_k$. Let I_π be the symmetric $k \times k$ matrix over $GF(2)$ with $(I_\pi)_{ij} = 1$ if and only if $\sigma_i = (ab)$ and $\sigma_j = (cd)$ with either $a < c < b < d$ or $c < a < d < b$.

Theorem 1 (*Cohn-Lempel equality*) *The number of orbits in $\{1, \dots, m\}$ under $\pi = \sigma\sigma_1\dots\sigma_k$ is $1 + \nu(I_\pi)$, where $\nu(I_\pi)$ is the $GF(2)$ -nullity of I_π .*

The Cohn-Lempel equality was reproven by Moran [21] and Stahl [24]. It was extended to non-disjoint transpositions $\sigma_1, \dots, \sigma_k$ by Beck and Moran [5, 6], who also pointed out that an equivalent equality was obtained much earlier by Brahana [8]. Other related results have been presented by Macris and Pulé [19], Lauri [18] and Jonsson [14].

Suppose D is a connected 2-in, 2-out digraph with $V(D) = \{v_1, \dots, v_n\}$ and $E(D) = \{e_1, \dots, e_m\}$; D may have loops or multiple edges. A *directed trail* in D is described by a sequence $v_{j_1} e_{j_1} v_{j_2} e_{j_2} \dots v_{j_c} e_{j_c} v_{j_{c+1}}$ of vertices and pairwise distinct edges such that each e_{j_k} is directed from v_{j_k} to $v_{j_{k+1}}$; a trail may also be described by its sequence of edges. If $v_{j_{c+1}} = v_{j_1}$ the trail is a *circuit*; the same circuit is described if the sequence is permuted cyclically, with the natural notation changes at the ends. D must have a directed Euler circuit, *i.e.*, a directed circuit that includes every edge. We presume the edges are indexed so that $e_1 \dots e_m$ is an Euler circuit, which we denote C . A partition P of $E(D)$ into directed circuits is associated to a permutation π_P of $\{1, \dots, m\}$, with $i\pi_P = j$ if e_i is followed immediately by e_j in one of the directed circuits of P . The elements of P correspond to the orbits in $\{1, \dots, m\}$ under π_P . P may also be specified by giving the subset $S_P \subseteq V(D)$ consisting of the vertices at which the incident circuit(s) of P do not follow the same edge-to-edge *transitions* as C [16]. If the edges directed into such a vertex v are e_{a-1} and e_{b-1} , and those directed outward are e_a and e_b , then saying that the incident circuit(s) of P do not follow the same transitions as C means that C is $e_{a-1}e_a \dots e_{b-1}e_b \dots$ and the incident circuit(s) of P are $e_{a-1}e_b \dots$ and $\dots e_{b-1}e_a$. (Obvious changes in indexing may be required if any of these edges is a loop or if $1 \in \{a, b\}$.) The permutation π_P is then of the form $\sigma\sigma_1 \dots \sigma_k$, with a transposition σ_i associated to each $v \in S_P$; if e_a and e_b are the edges directed outward from v then σ_i is (ab) .

Following [22], let $I(D, C)$ be the *interlace matrix* of D with respect to C : the $n \times n$ matrix over $GF(2)$ whose ij entry is 1 if and only if $i \neq j$ and v_i and v_j are *interlaced* in C , *i.e.*, when we follow C starting at v_i we encounter v_j , then v_i , then v_j again before finally returning to v_i . If P is a directed circuit partition of D then I_{π_P} is simply the submatrix of $I(D, C)$ that involves the rows and columns corresponding to elements of S_P .

Corollary 2 *Let D be a connected 2-in, 2-out digraph, and let $\mathcal{P}(D)$ be the set of partitions of $E(D)$ into directed circuits. Let $I(D, C)$ be the interlace matrix corresponding to an Euler circuit C of D , and for each subset $S \subseteq V(D)$ let $I_S(D, C)$ be the submatrix of $I(D, C)$ that involves the rows and columns corresponding to elements of S . Then*

$$\sum_{P \in \mathcal{P}(D)} (y-1)^{|P|-1} = \sum_{S \subseteq V(D)} (y-1)^{\nu(I_S(D, C))}.$$

Proof. $P \leftrightarrow S_P$ defines a one-to-one correspondence between elements of $\mathcal{P}(D)$ and subsets of $V(D)$, and the Cohn-Lempel equality tells us that for each $P \in \mathcal{P}(D)$, $|P| = \nu(I_{\pi_P}) + 1 = \nu(I_{S_P}(D, C)) + 1$. ■

Arratia, Bollobás, and Sorkin introduced the *interlace polynomials* of looped, undirected graphs in [2, 3, 4]. These invariants were first defined recursively, but soon it was shown that they are also given by formulas involving matrix nullities [1, 4]. Given an undirected graph G with $V(G) = \{v_1, \dots, v_n\}$, let $\mathcal{A}(G)$

be the $n \times n$ matrix with entries in $GF(2)$ given by $a_{ii} = 1$ if and only if v_i is looped, and for $i \neq j$, $a_{ij} = 1$ if and only if v_i and v_j are adjacent. For $S \subseteq V(G)$ let $\mathcal{A}(G)_S$ denote the submatrix of $\mathcal{A}(G)$ consisting of the rows and columns corresponding to elements of S ; equivalently $\mathcal{A}(G)_S = \mathcal{A}(G[S])$, where $G[S]$ denotes the subgraph of G induced by S .

Definition 3 *The vertex-nullity interlace polynomial of G is*

$$q_N(G) = \sum_{S \subseteq V(G)} (y - 1)^{\nu(\mathcal{A}(G)_S)}$$

and the (two-variable) interlace polynomial of G is

$$q(G) = \sum_{S \subseteq V(G)} (x - 1)^{|S| - \nu(\mathcal{A}(G)_S)} (y - 1)^{\nu(\mathcal{A}(G)_S)}.$$

Definition 3 may be applied to graphs with parallel edges or parallel loops, but parallels do not affect $\mathcal{A}(G)$ or the interlace polynomials.

Suppose D is a connected 2-in, 2-out digraph with an Euler circuit C , and H is the *interlace graph* of D with respect to C , *i.e.*, the undirected graph with $V(H) = V(D)$ and $\mathcal{A}(H) = I(D, C)$. Theorem 24 of [3] states that $q_N(H)$ is essentially the same as the generating function for partitions of $E(D)$ into directed circuits. The proof given there involves the recursive definition of q_N , but once it is recognized that q_N can also be given by Definition 3, it becomes clear that the relationship between $q_N(H)$ and directed circuit partitions of D is equivalent to Corollary 2 above.

The Kauffman bracket polynomial of a knot or link diagram (and other link invariants too) can be given by a sum whose terms are obtained by counting circuits in circuit partitions. As Arratia, Bollobás, and Sorkin observed in [3], this leads directly to a relationship between the Kauffman bracket and the vertex-nullity interlace polynomial. The fact that the Kauffman bracket can be described by formulas involving $GF(2)$ -nullity has also been noted by knot theorists; Soboleva [23] seems to have been the first to explicitly cite the Cohn-Lempel equality. Some of the formulas used by knot theorists resemble the Cohn-Lempel equality or Corollary 2 without being quite the same. For instance, Zulli [27] counted circuits using a formula that involves the $GF(2)$ -nullities of matrices that may have nonzero entries on the diagonal, and are all $n \times n$. More recently, Lando [17] and Mellor [20] used a formula that includes both the Cohn-Lempel equality and Zulli's formula. As is natural in the literature of knot theory, the discussions in these references are essentially topological – the arguments of Lando and Zulli involve the homology of surfaces, and Mellor and Soboleva are concerned with weight systems for link invariants – and they focus (implicitly or explicitly) on connected, planar digraphs.

In this note we present a combinatorial proof of an extended version of the Cohn-Lempel equality that applies to arbitrary circuit partitions in arbitrary

4-regular graphs. This extended Cohn-Lempel equality does not require that the 4-regular graph in question be connected, directed or planar, and it includes the various formulas just mentioned. The greater generality of the extended Cohn-Lempel equality is not only pleasing but also useful: it is a crucial part of an interlacement-based analysis of Kauffman's bracket for virtual links [15] developed by Zulli and the present author [25, 26], and as we see below it allows us to extend the relationship between circuit partitions and interlace polynomials to include interlace graphs that have had some loops attached.

Before stating the extended Cohn-Lempel equality we take a moment to establish notation and terminology. Suppose G is an undirected 4-regular graph. If G is connected it must have an Euler circuit C . Choose one of the two orientations of C , let D be the 2-in, 2-out digraph obtained from G by directing all edges according to that orientation, and let $I(D, C)$ be the interlace matrix of D with respect to C . If G is not connected then let C be a set of Euler circuits, one in each of the $c(G)$ connected components of G , and let D be a 2-in, 2-out digraph resulting from one of the $2^{c(G)}$ possible choices of orientations for the circuits in C . The interlace matrix $I(D, C)$ then consists of $c(G)$ diagonal blocks corresponding to the interlace matrices of the components of G with respect to the circuits of C ; the entries outside these diagonal blocks are all 0.

Let P be a partition of $E(G)$ into undirected circuits. Suppose $v_i \in V(G)$, and consider an edge e that is directed toward v_i in D . Some circuit of P must contain e . If we follow this circuit through v_i after traversing e , then there are three ways we might leave v_i : along the edge C uses to leave v_i after arriving along e , along the other edge directed away from v_i in D , or along the remaining edge directed toward v_i in D . We say P *follows C through v_i* in the first case, P is *orientation-consistent at v_i but does not follow C* in the second case, and P is *orientation-inconsistent at v_i* in the third case. Changing the choice of e or the orientations of the circuit(s) in C does not affect the descriptions of the three cases. (*N.B.* In order to provide well-defined descriptions of the three possibilities at looped vertices we should actually refer to *half-edges*; we leave this sharpening of terminology to the reader.) A matrix $I_P = I_P(D, C)$ is obtained from $I(D, C)$ as follows. If P follows C through v_i then the row and column of $I(D, C)$ corresponding to v_i are removed; if P is orientation-consistent at v_i but does not follow C then the row and column of $I(D, C)$ corresponding to v_i are retained without change; and if P is orientation-inconsistent at v_i then the row and column of $I(D, C)$ corresponding to v_i are retained with one change: their common diagonal entry is changed from 0 to 1.

Theorem 4 (*Extended Cohn-Lempel equality*) *If G is an undirected, 4-regular graph with $c(G)$ components and P is a partition of $E(G)$ into undirected circuits then*

$$|P| = \nu(I_P) + c(G).$$

As an example of the extended Cohn-Lempel equality consider the complete graph K_5 , with vertices denoted 1, 2, 3, 4 and 5. Let D be the directed version

of K_5 with edge-directions given by the Euler circuit $C = 1234513524$. If P follows C at vertex 1, is orientation-inconsistent at vertices 2 and 3, and is orientation-consistent but does not follow C at vertices 4 and 5 then

$$\nu(I_P(D, C)) = \nu \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = 0,$$

so $|P| = 1$. The one circuit in P is the Euler circuit 1254231534. The partition P' that disagrees with P only by following C at 3 corresponds to the matrix $I_{P'}(D, C)$ obtained by removing the second row and column of $I_P(D, C)$, so $\nu(I_{P'}(D, C)) = 1$. P' contains the circuits 1254234 and 135.

The extended Cohn-Lempel equality implies that the relationship between interlace polynomials and directed circuit partitions extends to looped interlace graphs.

Corollary 5 *Suppose C , D and G are as in Theorem 4, and H is obtained from the interlace graph of D with respect to C by attaching loops at some vertices. Then*

$$q_N(H) = \sum_{S \subseteq V(H)} (y - 1)^{|P_S| - c(G)},$$

where P_S is the undirected circuit partition that follows C at each vertex $v \notin S$, is orientation-inconsistent at each looped vertex $v \in S$, and is orientation-consistent but does not follow C at each unlooped vertex $v \in S$. Also, the two-variable interlace polynomial of H is

$$q(H) = \sum_{S \subseteq V(H)} (x - 1)^{|S| - |P_S| + c(G)} (y - 1)^{|P_S| - c(G)}.$$

In the balance of the paper we prove Theorem 4, derive an analogue of Corollary 5 for the multivariate interlace polynomial of Courcelle [10], and comment briefly on related results of Beck and Moran [5, 6], Macris and Pulé [19], Lauri [18] and Jonsson [14]. Before proceeding we should express our gratitude to D. P. Ilyutko and L. Zulli, whose discussions of [12] and [27] inspired this note. We are also grateful to Lafayette College for its support.

2 Proof of the extended Cohn-Lempel equality

The equality is proven under the assumption that G is connected; the general case follows as the contributions from different connected components are simply added together.

We begin with a special case: every entry of I_P is 0. This case falls under the original Cohn-Lempel equality but we provide an argument anyway, for the sake of completeness. If I_P is the empty matrix, then $P = \{C\}$ and the equality

is satisfied. If I_P is the 1×1 matrix (0) and the one entry corresponds to a , let aC_1a and aC_2a be circuits with $C = aC_1aC_2a$; then P consists of two separate circuits aC_1a and aC_2a , so the equality is satisfied. Proceeding by induction on the size of $I_P = \mathbf{0}$, let S_P be the set of vertices at which P does not follow C . Choose $a \in S_P$ so that $C = aC_1aC_2a$ with C_1 as short as possible. Then no element of S_P appears on C_1 , for a vertex that appears only once is interlaced with a (violating $I_P = \mathbf{0}$) and a vertex b that appears twice has $C = bC'_1bC'_2b$ with C'_1 shorter than C_1 (violating the choice of a). Let Q be the circuit partition that disagrees with P only by following C at a . Then I_Q is smaller than I_P , so the inductive hypothesis tells us that $|Q| = \nu(I_Q) + 1 = \nu(I_P)$. As $C = aC_1aC_2a$ and both P and Q follow C at every vertex of C_1 , it is clear that a appears on two circuits of P (aC_1a and another), these two circuits are united in Q , and the other elements of P and Q coincide. Hence $|P| = |Q| + 1 = \nu(I_P) + 1$.

If I_P is the 1×1 matrix (1) with a single entry corresponding to a , and $C = aC_1aC_2a$, then the equality is satisfied because P contains only the Euler circuit $aC_1a\bar{C}_2a$. Here \bar{C}_2 is the reverse of C_2 and the Euler circuit $aC_1a\bar{C}_2a$ is the κ -transform of C at a , denoted $C * a$ [7, 16].

The argument proceeds by induction on the size of $I_P \neq \mathbf{0}$. Suppose P is orientation-inconsistent with C at a vertex a , and let $C = aC_1aC_2a$. Then $C * a = aC_1a\bar{C}_2a$ is also an Euler circuit of G , and P follows $C * a$ through a . If $v \neq a$ is a vertex that appears on both C_1 and C_2 then either P follows both C and $C * a$ through v , or else P is orientation-inconsistent with respect to one of $C, C * a$ at v and orientation-consistent with the other of $C, C * a$ without following it through v . If $v \neq a$ is a vertex that appears on only one of C_1, C_2 then P has the same status with respect to C and $C * a$ at v . If $a \notin \{v, w\}$ and v and w both appear on C_1 and C_2 , then the interlacement of v and w with respect to $C * a$ is the opposite of their interlacement with respect to C . On the other hand, if $a \notin \{v, w\}$ and either v or w doesn't appear on both C_1 and C_2 then their interlacement with respect to $C * a$ is the same as their interlacement with respect to C . In sum, if $D * a$ denotes the digraph on G consistent with $C * a$ then

$$I_P(D, C) = \begin{pmatrix} 1 & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & M_{11} & M_{12} \\ \mathbf{0} & M_{21} & M_{22} \end{pmatrix} \quad \text{and} \quad I_P(D * a, C * a) = \begin{pmatrix} \bar{M}_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

for appropriate submatrices M_{ij} ; here \bar{M}_{11} differs from M_{11} in every entry. Adding the first row of $I_P(D, C)$ to each row involved in M_{11} and M_{12} , we see that $I_P(D * a, C * a)$ and $I_P(D, C)$ have the same nullity. As $I_P(D * a, C * a)$ is smaller than $I_P(D, C)$, induction tells us that $|P| = \nu(I_P(D, C)) + 1$.

Suppose now that there is no vertex at which P is orientation-inconsistent; this case too falls under the original Cohn-Lempel equality. As the equality has already been verified in the case $I_P = \mathbf{0}$, we presume that there are two interlaced vertices a and b such that P follows C neither at a nor at b . Let $C = aC_1bC_2aC_3bC_4a$, and $C * a * b * a = aC_1bC_4aC_3bC_2a$; then P follows

$C * a * b * a$ through both a and b . A case-by-case analysis shows that

$$I_P(D, C) = \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & M_{11} & M_{12} & M_{13} & M_{14} \\ \mathbf{1} & \mathbf{0} & M_{21} & M_{22} & M_{23} & M_{24} \\ \mathbf{0} & \mathbf{1} & M_{31} & M_{32} & M_{33} & M_{34} \\ \mathbf{0} & \mathbf{0} & M_{41} & M_{42} & M_{43} & M_{44} \end{pmatrix}$$

$$\text{and} \quad I_P(D, C * a * b * a) = \begin{pmatrix} M_{11} & \bar{M}_{12} & \bar{M}_{13} & M_{14} \\ \bar{M}_{21} & M_{22} & \bar{M}_{23} & M_{24} \\ \bar{M}_{31} & \bar{M}_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{pmatrix}$$

for appropriate submatrices. For instance, suppose v_i appears in C_2 and C_3 and v_j appears in C_2 and C_4 ; then v_i is interlaced with v_j with respect to C if and only if v_i precedes v_j in C_2 , whereas v_i is interlaced with v_j with respect to $C * a * b * a$ if and only if v_i follows v_j in C_2 . Consequently the entry of $I_P(D, C)$ corresponding to v_i and v_j , which falls in M_{13} and M_{31} , is the opposite of the corresponding entry of $I_P(D, C * a * b * a)$. Using elementary row operations we see that $I_P(D, C * a * b * a)$ and $I_P(D, C)$ have the same nullity; as $I_P(D, C * a * b * a)$ is smaller the inductive hypothesis tells us that $|P| = \nu(I_P(D, C)) + 1$.

Readers familiar with [1, 4] will recognize the matrix reductions in the argument. The same reductions are used to deduce the recursive properties of the interlace polynomials from their matrix formulas.

3 The multivariate interlace polynomial

Courcelle [10] introduced a multivariate interlace polynomial of a looped graph H , given by

$$C(H) = \sum_{\substack{A, B \subseteq V(H) \\ A \cap B = \emptyset}} \left(\prod_{a \in A} x_a \right) \left(\prod_{b \in B} y_b \right) u^{|A \cup B| - \nu((H \nabla B)[A \cup B])} v^{\nu((H \nabla B)[A \cup B])}$$

where $H \nabla B$ denotes the graph obtained from H by toggling loops at the vertices in B and u, v , and the various x_a and y_b are independent indeterminates. The contribution of each A, B pair to $C(H)$ is distinguished by the corresponding indeterminates. Consequently if D is a 2-in, 2-out digraph and H is a looped version of the interlace graph of D with respect to a set C of directed Euler circuits for the components of D , then the extended Cohn-Lempel equality tells us that $C(H)$ is essentially the same as the list of all partitions of $E(D)$ into undirected circuits, with each partition listed along with its cardinality. That is, $C(H)$ is essentially the *transition polynomial* studied by Jaeger [13] and Ellis-Monaghan and Sarmiento [11].

Corollary 6 *Suppose D is a 2-in, 2-out digraph, C contains a directed Euler circuit for each component of D , and H is obtained from the interlace graph of D with respect to C by attaching loops at some vertices. Then the multivariate interlace polynomial of H is*

$$C(H) = \sum_{\substack{A, B \subseteq V(H) \\ A \cap B = \emptyset}} \left(\prod_{a \in A} x_a u \right) \left(\prod_{b \in B} y_b u \right) \left(\frac{v}{u} \right)^{|P_{A,B}| - c(D)}$$

where $P_{A,B}$ is the undirected circuit partition that follows C at vertices not in $A \cup B$, is orientation-inconsistent at looped vertices in A and unlooped vertices in B , and is orientation-consistent but does not follow C at unlooped vertices in A and looped vertices in B .

Proof. Reformulating the definition,

$$C(H) = \sum_{\substack{A, B \subseteq V(H) \\ A \cap B = \emptyset}} \left(\prod_{a \in A} x_a u \right) \left(\prod_{b \in B} y_b u \right) \left(\frac{v}{u} \right)^{\nu((H \nabla B)[A \cup B])}.$$

■

4 Two remarks

1. The original form of the Cohn-Lempel equality is not completely general. For instance, if $n \geq 3$ then the identity permutation is not $\sigma\sigma_1 \dots \sigma_k$ for any disjoint transpositions $\sigma_1, \dots, \sigma_k$. Beck and Moran [5, 6] extended the Cohn-Lempel equality to arbitrary permutations by removing the requirement that the σ_i be disjoint. Theorem 4 may also be applied to arbitrary permutations. If π is a permutation of $\{1, \dots, 2n\}$ choose any partition of $\{1, \dots, 2n\}$ into pairs, and construct the directed graph D whose n vertices correspond to these pairs and whose $2n$ edges correspond to $1, \dots, 2n$, with the edge corresponding to i directed from the vertex corresponding to the pair containing i to the vertex corresponding to the pair containing $i\pi$. If π' is a permutation of $\{1, \dots, 2n-1\}$, first replace it with the permutation π of $\{1, \dots, 2n\}$ that has $i\pi = i\pi'$ for $i < 2n-1$, $(2n-1)\pi = 2n$, and $(2n)\pi = (2n-1)\pi'$; then construct D as before.

2. Macris and Pulé [19], Lauri [18] and Jonsson [14] introduced skew-symmetric integer matrices that reduce (mod 2) to I_π and whose nullity over the rationals is $\nu(I_\pi)$. In general, however, there is no skew-symmetric version of $I_P(D, C)$ whose \mathbb{Q} -nullity can be used in Theorem 4. For example, consider the directed graph D with vertices denoted $1, 2, 3 \pmod{3}$ in which there are two edges directed from vertex i to vertex $i+1$ for each i . $E(D)$ has a partition P containing three undirected circuits; each element of P consists of two parallel edges. $I_P(D, C)$ is the 3×3 binary matrix with every entry 1, and $\nu(I_P(D, C)) = 2$ in accordance with the extended Cohn-Lempel equality. However a skew-symmetric 3×3 matrix of \mathbb{Q} -nullity 2 must have at least five of its nine entries equal to 0.

5 Dedication

T. H. Brylawski's work has influenced a generation of researchers studying matroids and the Tutte polynomial. My training in knot theory was focused on algebraic topology rather than combinatorics, so I particularly appreciated the clarity and thoroughness of his expository writing. No less important was his professional hospitality, which made me feel welcome in a new field. This note is gratefully dedicated to his memory.

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